Optimisation

For a given design requirement many possible solutions may exist. The design task is to seek the best trade-off or compromise between often conflicting requirements and constraints. The process of determining the best solution is called optimisation. There are many techniques available to the engineer including differentiation, Lagrange multipliers and linear programming.

1- Differentiation

For a multiple independent variable function, take partial derivatives with respect to each variable and equate the resulting equations to zero; e.g. for the function $f(x_1, x_2, x_3, \ldots, x_n)$, taking derivatives with respect to each variable in turn gives

$$\frac{\partial f(x_1, x_2, x_3, \ldots, x_n)}{\partial x_1} = 0 \qquad \ldots\ldots\ldots(1)$$

$$\frac{\partial f(x_1, x_2, x_3, \ldots, x_n)}{\partial x_2} = 0 \qquad \ldots\ldots\ldots(2)$$

$$\frac{\partial f(x_1, x_2, x_3, \ldots, x_n)}{\partial x_n} = 0 \qquad \ldots\ldots\ldots(3)$$

The next step is to solve the resulting equations simultaneously.

Example: The revenue (الايرادات) from selling components is given by $r(x) = 9x$ and the cost (الكلفة) of production and marketing is given by $c(x) = x^3 - 6x^2 + 15x$. What, if any, is the level of production which maximises profit?

(ما مستوى الانتاج الذي يحقق اكبر ربح؟)

Solution: The profit is given by $p(x) = r(x) = c(x)$

$$p = 9x - x^3 + 6x^2 - 15x = -x^3 + 6x^2 - 6x$$

$$\frac{\partial p}{\partial x} = -3x^2 + 12x - 6$$

$$\text{Setting } \frac{dp}{dx} = 0 \text{ gives } -3x^2 + 12x - 6 = 0$$

$$\text{or} \qquad x^2 - 4x + 2 = 0$$

Solving for x gives $x = 2 \pm \sqrt{2}$

The solution represents a local maximum given by $2 + \sqrt{2}$ and a minimum given by $2 - \sqrt{2}$; i.e. the profit is a maximum when $x = 2 + \sqrt{2}$

2- Lagrange multipliers

To find the stationary points of the function $u = f(x, y, z)$ subject to the constraints $\phi(x, y, z) = 0$ determine the differentials:

$$\left.\begin{aligned} \frac{\partial u}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0 \\ \frac{\partial u}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0 \\ \frac{\partial u}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0 \end{aligned}\right\} \ldots\ldots\ldots(4)$$

The solution of these equations will give values for x, y and z.

Example: Determine the total height and diameter: of a storage tank, comprising a hemispherical top and cylindrical body, if the container holds 500 m^3 of liquid and the surface area is to be minimised.

Solution The surface area is given by

$$A = 3\pi r^2 + 2\pi h r$$

The volume of the tank is given by

$$V = \pi r^2 h + \frac{2}{3}\pi r^3 = 500$$

$$\pi r^2 h + \frac{2}{3}\pi r^3 - 500 = 0 \text{ (constraint equation)}$$

We need to find:

$$\frac{\partial A}{\partial r} + \lambda \frac{\partial V}{\partial r} = 0 \quad \text{and} \quad \frac{\partial A}{\partial h} + \lambda \frac{\partial V}{\partial h} = 0$$

$$\frac{\partial A}{\partial r} = 6\pi r + 2\pi h \quad , \quad \frac{\partial V}{\partial r} = 2\pi h r + 2\pi r^2$$

$$\frac{\partial A}{\partial h} = 2\pi r \quad , \quad \frac{\partial V}{\partial h} = \pi r^2$$

$$6\pi r + 2\pi h + \lambda(2\pi h r + 2\pi r^2) = 0$$

$$2\pi r + \lambda \pi r^2 = 0 \quad , \quad \lambda = -2/r$$

$$6\pi r + 2\pi h - 4\pi h - 4\pi r = 0 \quad \rightarrow \quad r = h$$

From the constraint equation

$$\pi r^3 + \frac{2}{3}\pi r^3 = \frac{5}{3}\pi r^3 = 500,$$

$$r = \left(\frac{1500}{5\pi}\right)^{\frac{1}{3}} = 4.57 \text{ m} \qquad [\text{Ans.}]$$

$$\text{height } H = h + r = 9.14 \text{ m.} \qquad [\text{Ans.}]$$